ON THE EXPANSION OF sn x.

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(1.) Since sn $(-x) = -$ sn x, we may assume

$$\begin{aligned}
\operatorname{sn} x &= A_0 x + A_1 x^3 + A_2 x^5 + \ldots ;\\
\therefore\ \operatorname{sn} y &= A_0 y + A_1 y^3 + A_2 y^5 + \ldots ,\\
\operatorname{sn}(x+y) &= A_0 (x+y) + A_1 (x+y)^3 + \ldots ,\\
\operatorname{cn} x \operatorname{dn} x = \operatorname{sn}' x &= A_0 + 3A_1 x^2 + 5A_2 x^4 + \ldots ;
\end{aligned}$$

and so for cn y dn y.

Substituting these developments in the formula

$$\operatorname{sn}(x+y) = \frac{\operatorname{sn}^2 x - \operatorname{sn}^2 y}{\operatorname{sn} x \operatorname{cn} y \operatorname{dn} y - \operatorname{sn} y \operatorname{cn} x \operatorname{dn} x},$$

we have, after clearing of fractions,

$$\begin{aligned}
&[A_0 (x+y) + A_1 (x+y)^3 + \ldots]\\
&\quad \times \left[\begin{array}{l} \ \ (A_0 x + A_1 x^3 + \ldots)(A_0 + 3A_1 y^2 + \ldots)\\ -(A_0 y + A_1 y^3 + \ldots)(A_0 + 3A_1 x^2 + \ldots)\end{array}\right]\\
&\quad = (A_0 x + A_1 x^3 + \ldots)^2 - (A_0 y + A_1 y^3 + \ldots)^2. \qquad (a)
\end{aligned}$$

Since the second member contains terms of the form cx^n or cy^n only, it follows that, if the first member be expanded and similar terms united, the coefficient of any resulting term, whose form is $cx^p y^q$ where $p, q > 0$, must be equal to zero. We could thus obtain relations between the coefficients $A_0, A_1, \ldots A_n$. But actual multiplication is not necessary, since we are not concerned with the entire product. We proceed as follows:—

(2.) Performing the multiplication indicated in the first term of the second factor of the first member of (a), we have

$$\begin{aligned}
&(A_0 x + A_1 x^3 + \ldots)(A_0 + 3A_1 y^2 + \ldots)\\
&\qquad = A_0 A_0 y^0 x + A_0 A_1 y^0 x^3 + A_0 A_2 y^0 x^5 + \ldots\\
&\qquad + 3A_1 A_0 y^2 x + 3A_1 A_1 y^2 x^3 + 3A_1 A_2 y^2 x^5 + \ldots\\
&\qquad + 5A_2 A_0 y^4 x + 5A_2 A_1 y^4 x^3 + 5A_2 A_2 y^4 x^5 + \ldots\\
&\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

$=$

Order in y.		Order in x. 1	3	5	7	
	0	A_0A_0	A_0A_1	A_0A_2	A_0A_3	. . .
	2	$3A_1A_0$	$3A_1A_1$	$3A_1A_2$	$3A_1A_3$	. . .
	4	$5A_2A_0$	$5A_2A_1$	$5A_2A_2$	$5A_2A_3$	. . .
	6	$7A_3A_0$	$7A_3A_1$	$7A_3A_2$	$7A_3A_3$	. . .
		. . .	. . .	. . .	. . .	. . .

(β)

(3.) Because of symmetry, if x and y of this table be interchanged, we have the product of $A_0y + A_1y^3 + \ldots$ and $A_0 + 3A_1x^2 + \ldots$.

(4.) The first factor of (α) may be written $\sum\limits_{n=0}^{n=\infty} A_n(x+y)^{2n+1}$,

or

	0	1	2	3	4	5	6	7
0		A_0		A_1		A_2		A_3
1	A_0		$3A_1$		$5A_2$		$7A_3$	
2		$3A_1$		$10A_2$		$21A_3$		
3	A_1		$10A_2$		$35A_3$			
4		$5A_2$		. . .				
5	A_2		. . .					
6		. . .						
7	. . .							

(γ)

Suppose that we multiply (β) by (γ) and wish to find all terms in which x and y have the exponents p and q, respectively. Each of such terms will consist of two factors $c_{st}x^sy^t$, $c_{uv}x^uy^v$, the one from (β), the other from (γ), where $s + u = p$ and $t + v = q$. Since $s \overline{\gtrless} p$ and $t \overline{\gtrless} q$, we see at once, what factors can come from (β); and since $p = s + u$ and $q = t + v$, the factors from (γ) are determined. In fact, we have only to multiply the first term of (β) by that term of (γ) whose order

is the highest permissible; again, the next higher in (β) with reference to either variable by the next lower in (γ) with reference to the same variable, and so continue until s and t have reached their limits.

(5.) EXAMPLE. Let it be required to find the entire coefficient of x^5y^3.

From what has just been said, (β) multiplied by (γ) would give as the coeffient of x^5y^3

$$\begin{aligned} &A_0A_0 \,.\, 35A_3 + A_0A_1 \,.\, 10A_2 + A_0A_2 \,.\, A_1 \\ + &3A_1A_0 \,.\, 5A_2 + 3A_1A_1 \,.\, 3A_1 + 3A_1A_2 \,.\, A_0. \end{aligned}$$

Now interchange x and y; there results as the coefficient of y^5x^3

$$\begin{aligned} &A_0A_0 \,.\, 21A_3 + A_0A_1 \,.\, A_2 \\ + &3A_1A_0 \,.\, 10A_2 + 3A_1A_1 \,.\, A_1 \\ + &5A_2A_0 \,.\, 3A_1 + 5A_2A_1 \,.\, A_0. \end{aligned}$$

But these must be equal, since the entire coefficient of x^5y^3 equals zero [(3), (4)]. Collecting and transposing, we have

$$7A_3A_0^2 - 11A_2A_1A_0 + 3A_1^3 = 0.$$

From other considerations, we know that $A_0 = 1$;

$$\therefore\; 7A_3 - 11A_2A_1 + 3A_1^3 = 0.$$

(6.) From (γ), it is evident that we can always bring A_m into an equation involving A's of lower subscript only, if we take $p + q = 2\,(m + 1)$. For example, we could have obtained the above equation by using x^6y^2 instead of x^5y^3. Obviously, the smaller we can take q, the more simple will be the operation.

If $q = 1$, the result is an identity, and so of no value to us; but if $q = 2$, the result in general will not be an identity. We observe that the coefficient of the A_m, $q = 2$, from (γ) is the same as the coefficient of the third term of $(a + b)^{2m+1}$, or $m\,(2m + 1)$. In (β), A_m does not occur while we are considering x^py^2. Now interchange x and y; i. e. consider x^2y^p. The A_m from (γ) has the same coefficient as the second term of $(a + b)^{2m+1}$, or $2m + 1$; from (β), it also has the coefficient $2m + 1$. These added together give $2\,(2m + 1)$; but $2\,(2m + 1)$ is not equal to $m\,(2m + 1)$, except for $m = 2$. Since, in general, we can obtain the required relations by making $q = 2$, we shall hereafter confine ourselves to that value. The operation then becomes very simple. E.g. Required the value of A_4.

Here $p + q = 10$; therefore $p = 8$. We have, then,

$$\begin{aligned} &A_0\,(A_0 \,.\, 36A_4 + A_1 \,.\, 21A_3 + A_2 \,.\, 10A_2 + A_3 \,.\, 3A_1)\;. \\ + &3A_1\,(A_0 \,.\, A_3 + A_1 \,.\, A_2 + A_2 \,.\, A_1 + A_3 \,.\, A_0) \\ = &A_0\,(A_0 \,.\, 9A_4 + 3A_1 \,.\, 7A_3 + 5A_2 \,.\, 5A_2 + 7A_3 \,.\, 3A_1 + 9A_4 \,.\, A_0); \end{aligned}$$

$$\therefore\; 6A_4 - 4A_3A_1 - 5A_2^2 + 2A_2A_1^2 = 0.$$

(7.) The relation sought is embodied in the following general equation:—

$$A_0 \sum_{r=1}^{r=m} r(2r+1) A_r A_{m-r} + 3A_1 \sum_{r=1}^{r=m} A_{r-1} A_{m-r}$$
$$= A_0 \sum_{r=1}^{r=m+1} (2r-1)(2m-2r+3) A_{r-1} A_{m-r+1};$$

$$\therefore A_m = \frac{\left\{ \begin{array}{l} A_0 \displaystyle\sum_{r=2}^{r=m+1} (2r-1)(2m-2r+3) A_{r-1} A_{m-r+1} \\ -3A_1 \displaystyle\sum_{r=1}^{r=m} A_{r-1} A_{m-r} - A_0 \displaystyle\sum_{r=1}^{r=m-1} r(2r+1) A_r A_{m-r} \end{array} \right\}}{(m-2)(2m+1) A_0^2}.$$

m odd

$$A_m = \frac{\left\{ \begin{array}{l} 2A_0 \displaystyle\sum_{r=2}^{r=\frac{1}{2}(m+1)} (2r-1)(2m-2r+3) A_{r-1} A_{m-r+1} \\ -6A_1 \displaystyle\sum_{r=1}^{r=\frac{1}{2}(m-1)} A_{r-1} A_{m-r} - 3A_1 A^2_{\frac{1}{2}(m-1)} - \displaystyle\sum_{r=1}^{r=m-1} r(2r+1) A_r A_{m-r} \end{array} \right\}}{(m-2)(2m+1) A_0^2}.$$

m even

$$A_m = \frac{\left\{ \begin{array}{l} 2A_0 \displaystyle\sum_{r=2}^{r=\frac{1}{2}m} (2r-1)(2m-2r+3) A_{r-1} A_{m-r+1} + (m+1)^2 A_0 A^2_{\frac{1}{2}m} \\ -6A_1 \displaystyle\sum_{r=1}^{r=\frac{1}{2}m} A_{r-1} A_{m-r} - A_0 \displaystyle\sum_{r=1}^{r=m-1} r(2r+1) A_r A_{m-r} \end{array} \right\}}{(m-2)(2m+1) A_0^2}.$$

By giving m the values 3, 4, . . . , we have

$$\begin{aligned}
&7A_3 - 11A_2A_1 + 3A_1^3 = 0, \\
&6A_4 - 4A_3A_1 - 5A_2^2 + 2A_2A_1^2 = 0, \\
&11A_5 - 3A_4A_1 - 13A_3A_2 + 2A_3A_1^2 + A_2^2A_1 = 0, \\
&26A_6 - A_5A_1 - 22A_4A_2 + 3A_4A_1^2 - 14A_3^2 + 3A_3A_2A_1 = 0.
\end{aligned}$$

. . . .

OBSERVATIONS.

All terms are of order three, counting the A_0's, and the weight of each term of any of these equations equals the greatest subscript, equals the number of terms.

A_1 and A_2 are supposed to be known; they may be called the *source* of all succeeding ones. For in terms of these two, any that follow may be expressed.

NOTE.—A similar method is applicable to cn x and dn x.